1 Jan 23, 2023

1.1 Definition of a Game

Definition 1: A

trategic game consists of

- a set of players $\{1, ..., n\}$,
- for each player i, a set of actions A_i,
- for each player i, a payoff function $u_i : A_1 \times \cdots \times A_n \to \mathbb{R}$ which assigns a preference to every action profile.

Example 1. If the game has two players, Anya and Loid, then we can assign Anya to be player 1, and Loid to be player 2. So $n = 2$.

- For player 1 (player A), her action set maybe $A_1 = \{\text{sleeps}, \text{studies}\}$. For player 2, his action set maybe $A_2 = \{\text{goes out}, \text{stays at home}\}$.
- An action profile records one possible situation of the game. For example,

$$\text{Anya sleeps and Loid goes out } = \underbrace{\{\text{sleeps}, \text{goes out}\}}_{\text{An action profile}}$$

  - Note that in here we have assumed that the players make decisions at the same time.
  - Then the set $F := A_1 \times A_2 = (x, y) : x \in A_1, y \in A_2$ contains every possible outcomes of this game.
- In reality we do not compare numbers in our mind when we make most decisions. But in Game Theory, we need to assign a number to each action profile. This number is called the payoff. This is done by defining a payoff function $u_1 : F \to \mathbb{R}$ and $u_2 : F \to \mathbb{R}$.
- For more examples, please read the lecture notes.

□

1.2 Nash Equilibrium

John Nash (1928-2015) received the Nobel Economic Sciences Prize, among many other awards, for his work in game theory, more specifically the Nash equilibrium theory. You may read more about him here: https://www.theguardian.com/science/alexs-adventures-in-numberland/2015/may/24/john-nashs-unique-approach-produced-quantum-leaps-in-economics-and-maths.

Definition 2: Nash Equilibrium (NE)

In a strategic game with ordinal preferences, a Nash equilibrium (NE) is an **action profile** in which no player wants to change their action, provided **nobody else changes action**.

- In the previous example, if the current situation is in a NE, and Anya changes her action, then Anya's payoff will not increase. (Her payoff may be the same, or may decrease.)
- Here is an interesting video about NE: https://www.youtube.com/watch?v=jILgxeNBK_8.
- We can translate the idea of NE mathematically.

Definition 3: Mathematically representation of NE

An action profile a^* is a NE if for all i and $a_i^{'} \in A_i$, we have $u_i(a^*) \geq u_i(a_i^{'}, a^*)$

- In here,

$$a_i^{'}, a^*$$

means the action profile a^* with the i-th action replace by $a_i^{'}$.

- How do we find the NE (for some simple games)? List all action profiles, and check which action profile(s) satisfies the requirement of NE. There are many examples in the lecture notes.
- In particular, Example 3.3 finds all NE for the Prisoner's Dilemma. We can see that the NE does not give the best outcome for both players. NE is (T,T) which gives both players the payoff 1. The best possible outcome is (Q,Q) which gives both players the payoff 3. Why?

1.3 Best Response

Given a game with n players and an action profile $a = (a_1, \cdots, a_n)$. We then denote $a_{-i} = (a_1, \cdots, a_i - 1, a_i + 1, \cdots, a_n)$. That is the action profile $a = (a_1, \cdots, a_n)$ with the action of player i deleted. Similarly, let $A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n := A_{-i}$.

Definition 4: C

nsider a strategic game with ordinal preferences with n players, each with an action set A_i and payoff function u_i. The **best response** for player i is a function

$$B_i : A_{-i} \to \mathcal{P}(A_i) \tag{1}$$

such that

$$B_i(a_{-i}) = \{a_j \in A_i : u_i(a_j, a_{-i}) \geq u_i(a_j^{'}, a_{-i}) \ \forall a_j' \in A_i\}. \tag{2}$$

What does the best response gives when we input a set a_{-i}?

- Notice that the best response is a function from sets to sets.
- The range is the subset of possible action of player i.
- When we input a_{-i}, the output of the function maybe $\{a_1, a_3, a_5, a_6, a_9\}$ or possibly be $\varnothing$.
- The output must satisfy the condition

$$u_i(a_i, a_{-i}) \geq u_i(a_{i'}, a_{-i}). \tag{3}$$

  For example, if a_5 is in the output then and $n = 7$,

$$\begin{aligned} u_i(a_5, a_{-i}) &\geq u_i(a_1, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_2, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_3, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_4, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_5, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_6, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_7, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_8, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_9, a_{-i}). \end{aligned} \tag{4}$$

  If there are totally 9 actions that player i can use.
- Thus the action a_5 is the best response that player i can do given that others use a_{-i}.

2 Jan 30, 2023

2.1 Nash Equilibrium

John Nash (1928-2015) received the Nobel Economic Sciences Prize, among many other awards, for his work in game theory, more specifically the Nash equilibrium theory. You may read more about him here: https://www.theguardian.com/science/alexs-adventures-in-numberland/2015/may/24/john-nashs-unique-approach-produced-quantum-leaps-in-economics-and-maths.

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- In here,
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means the action profile a^* with the i-th action replace by $a_i^{'}$.
- How do we find the NE (for some simple games)? List all action profiles, and check which action profile(s) satisfies the requirement of NE. There are many examples in the lecture notes.

Example 2. Consider this Prisoner's Dilemma

$$\begin{array}{ccc} & Q & T \\ Q & (3,3) & (0,4) \\ T & (4,0) & (1,1) \end{array} \tag{5}$$

The action profiles for this game are: (Q,Q), (Q,T), (T,Q), (T,T). What is the NE for this game? What can you say about the NE? □

2.2 Best Response

Given a game with n players and an action profile $a = (a_1, \cdots, a_n)$. We then denote $a_{-i} = (a_1, \cdots, a_i - 1, a_i + 1, \cdots, a_n)$. That is the action profile $a = (a_1, \cdots, a_n)$ with the action of player i deleted. Similarly, let $A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n := A_{-i}$.

Definition 7: C

nsider a strategic game with ordinal preferences with n players, each with an action set A_i and payoff function u_i. The **best response** for player i is a function

$$B_i : A_{-i} \to \mathcal{P}(A_i) \tag{6}$$

such that

$$B_i(a_{-i}) = \{a_j \in A_i : u_i(a_j, a_{-i}) \geq u_i(a_j^{'}, a_{-i}) \ \forall a_j' \in A_i\}. \tag{7}$$

What does the best response gives when we input a set a_{-i}?

- Notice that the best response is a function from sets to sets.
- The range is the subset of possible action of player i.
- When we input a_{-i}, the output of the function maybe $\{a_1, a_3, a_5, a_6, a_9\}$ or possibly be $\varnothing$.
- The output must satisfy the condition

  $$u_i(a_i, a_{-i}) \geq u_i(a_{i'}, a_{-i}). \tag{8}$$

  For example, if a_5 is in the output then and $n = 7$,

  $$\begin{aligned} u_i(a_5, a_{-i}) &\geq u_i(a_1, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_2, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_3, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_4, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_5, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_6, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_7, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_8, a_{-i}), \\ u_i(a_5, a_{-i}) &\geq u_i(a_9, a_{-i}). \end{aligned} \tag{9}$$

  If there are totally 9 actions that player i can use.
- Thus the action a_5 is the best response that player i can do given that others use a_{-i}.

Example 3. Find the Nash equilibria of the two-player strategic game in which each player's set of actions is the set of nonnegative numbers and the players' payoff functions are

$$u_1(a_1, a_2) = a_1(a_2 - a_1), \quad u_2(a_1, a_2) = a_2(1 - a_1 - a_2). \tag{10}$$

Solution: We can take the derivative of u_1 with respect to a_1 and the derivative of u_2 with respect to a_2 to obtain

$$\begin{aligned} \frac{\partial a_1}{\partial a_1} &= a_2 - 2a_1, \\ \frac{\partial a_2}{\partial a_2} &= 1 - a_1 - 2a_2. \end{aligned} \tag{11}$$

Setting the two derivative equals to zero. We get a system of equation in a_1 and a_2

$$\begin{cases} a_1 = \frac{a_2}{2} \\ a_2 = \frac{1-a_1}{2} \end{cases} \tag{12}$$

The solution is $a_1 = \frac{1}{5}$ and $a_2 = \frac{2}{5}$. This is the NE. We can quickly check our answer by considering the behavior of the payoff function for fixed a_1 and a_2. $\square$

3 Feb 6, 2023

3.1 Dominated Actions

Sometimes the payoff matrix is quite special. Consider a matrix like this. Denote a_i to be the i-th row. Let

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \tag{13}$$

where $a_1 > a_2 > a_3$ when we are comparing the rows entry-wise. Consider that you are player 1, that is, you are the row player, and you switch strategies by switching rows. In this case, what would you do to maximize your payoff? You would just pick the first row as your action. Because no matter what happened. You will receive the highest possible payoff.

Definition 8: Dominated Actions

Player i's action $a_i^{''}$ strictly dominates his action $a_i^{'}$ if for for all action profiles a_{-i} in A_{-i},

$$u_i(a_i^{''}, a_{-i}) > u_i(a_i^{'}, a_{-i}). \tag{14}$$

Suppose that you are player i. Essentially, Definition (8) said that when you pick $a_i^{''}$, then no matter what the others do. You would still get the best that you can get.

If the Nash Equilibrium is a "solution" to the game for everyone. Then the dominated action is a "solution" to player i. *What is the connection between Nash Equilibrium and dominated actions?*

Example 4. Consider the following payoff matrix

	b_1	b_2	b_3
a_1	(10,2)	(1,2)	(5,5)
a_1	(10,3)	(6,2)	(9,9)
a_1	(10,1)	(3,1)	(7,2)

(15)

Can you find out which action dominates which actions? □

3.2 Auctions

In an auction, players bid on some item and the item is sold to the highest bidder (not necessarily at the price of the highest bid). Generally, bids are either called out sequentially or may be submitted in sealed envelopes. We will consider sealed-bid auctions which we can formulate as a strategic game.

Side Story

There are many types of auctions. For the simplest case, there are English and Dutch auctions, the item's price goes up (English) or down (Dutch) and everyone knows at all times which bidders are interested in the item.

Paul Milgrom and Robert Wilson, have studied how auctions work. They have also used their insights to design new auction formats for goods and services that are difficult to sell in a traditional way, such as radio frequencies. Their discoveries have benefitted sellers, buyers, and taxpayers around the world.

Robert Wilson and Paul Milgrom on the morning of the day they received the 2020 Nobel Memorial Prize in Economic Sciences.

3.3 Defining a sealed-bid second-price auction

First let us assume that:

- There are n players.
- Player i has a number called valuation, denoted by v_i. (The item is worth v_i to player i.)
- $v_1 > v_2 > \cdots > v_n > 0$.
- $v_i \neq v_j$ for $i \neq j$.

A game requires action sets A_i for each player i. The action player i can use is to place a bid of $b_i > 0$. To break a possible tie, we assume that the player with the smaller index wins the auction, that is if $b_3 = b_5$ then player 3 wins. We assume each player knows the valuation of every player.

Definition 9: Payoff function for sealed-bid second-price auction

The sealed-bid second-price auction has the payoff function

$$u_i(b_1, \cdots b_n) = \begin{cases} v_i - \hat{b}, & \text{when } b_i > b_j \text{ for all } j < i \text{ and } b_i \geq b_j \text{ for all } j > i, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

where $\hat{b} = \max\{b_j : j \neq i\}$.

Let us pause here to inspect this payoff function. We note some observations here.

- u_i is the payoff, thus player i would want it to be as large as possible.

- There should be only one payoff function u_i that has no-zero value as by the way we designed our auction, there is only one winner. All other payoff functions would have zero payoffs.
- In the most ideal case, the winner player i should pay very little, for example, to place a bid of \$1 and win. Then his payoff $v_i - 1$.

Next, let us see an example.
Example 5. Consider $n = 3$, each player with valuation v_i.

1. What is the outcome of the action profile $(b_1, b_2, b_3) = (1, 1, 3)$?
2. What is the outcome of the action profile $(b_1, b_2, b_3) = (20, 20, 1)$?
3. What is the outcome of the action profile $(b_1, b_2, b_3) = (v_1, v_2, v_3)$?

□

3.4 Homework 1 #5

In Homework 1 #5, the question is here.

> Two candidates Alice and Bob, compete in an election Of the n citizens, k support Alice and $m = n - k$ support Bob. Each citizen decides whether to vote at a cost, for the candidate they support, or to abstain A citizen who abstains receives the payoff of 2 if the candidate they support wins, a payoff of 1 if this candidate ties for first place, and a payoff of 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases where $0 < c < 1$.
>
> (a) For $k = m = 1$, is this game equivalent to any other game considered so far?
>
> (b) For k = m, find the set of Nash equilibria. (Is the action profile in which everyone votes a Nash equilibrium? Is there any Nash equilibrium in which the candidates tie and not everyone votes? Is there any Nash equilibrium in which one of the candidates wins by one vote? Is there any Nash equilibrium in which one of the candidates wins by two or more votes?)

(a). The payoff matrix is

$$\begin{array}{r c} & \begin{array}{cc} \text{Vote} & \text{Don't Vote} \end{array} \\ \begin{array}{r} \text{Vote} \\ \text{Don't Vote} \end{array} & \begin{bmatrix} (1-c, 1-c) & (2-c, 0) \\ (0, 2-c) & (1, 1) \end{bmatrix} \end{array}. \tag{17}$$

To decide what is the game equivalent to, the numerical values aren't so important. We should first consider the what is the player's action in different cases. Let us compare it with a payoff matrix of a Prisoner dilemma

$$\begin{bmatrix} (3,3) & (1,4) \\ (4,1) & (2,2) \end{bmatrix}. \tag{18}$$

(b). Again a NE is an action profile. Let us look at the professor's solution.

1. Suppose everyone votes, then the result is a tie and everyone has a payoff of $1 - c$. If any player chooses to change their action and does not vote, then their candidate loses and their payoff becomes 0. This is an NE (it is the only NE).

2. Suppose $l < n$ people vote and the result is a tie. If a non-voter decides to change their action, then their candidate wins and their payoff increases from 1 to $2 - c$. This is not an NE.

3. Suppose a candidate wins by exactly one vote. If a non-voter for the losing party decides to change their action, then their candidate ties and their payoff increases from 0 to $1 - c$. This is not an NE.

4. Suppose a candidate wins by at least two votes. If a voter for the winning party decides to change their action, then their candidate still wins and their payoff increases from $2 - c$ to 2. This is not an NE.

These are the possible action profiles. (WHY?)

4 Feb 27, 2023

Today we will talk about Midterm problem 1 and problem 5 and the Extensive Games.

4.1 Extensive Games

In the previous examples, players make their decision at the same time. In extensive games, players make their decision turn by turn. For example, playing chess is an extensive game.

Definition 10: Extensive Games

An extensive game consists of four components:

- a set of n players,
- a set of terminal histories – a history is a sequence of actions over part of the game,
- player function P (whose turn is it after any history/after each action),
- preferences, represented by a payoff function u_i for each player, assigning a value to each terminal history.

We can solve the extensive games by *backward induction* and the outcome obtained from the backward induction is called a *subgame perfect Nash Equilibrium*.

A **strategy** in an extensive game is a similar concept to a strategy in a normal form game.

Definition 11

A strategy s_i for player i is a function that assigns one action to each history where it is their turn.

With strategy defined, we may further consider what kind of strategy the player makes. We have the so-called *strategy profile*

Definition 12

A strategy profile s is a choice of strategy for each player, e.g. $s = (s_1, s_2)$. Given a strategy profile, we can determine how an extensive game will play out.

The SPNE strategy profile is a strategy profile $s = (s_1, s_2)$, such that s_1 is the strategy chosen in the backward induction by player 1 and s_2 is the strategy chosen in the backward induction by player 2.

5 Mar 6, 2023

5.1 Probability

- A random variable: a variable that is random.
- Probability: A number within $[0, 1] \subset \mathbb{R}$.
- An event: a set of outcomes from some random variables.

Definition 13

Let $E \subset \Omega$ be any event. The probability of E is

$$P(E) = \sum_{\omega \in E} m(\omega). \tag{19}$$

In this definition, m is the distribution function.

Example 6. Given a fair dice. What is the random variable that represents the dice? Give one example of an event and find the probability of that event.

Solution: The random variable is of the dice is just X where $X = 1/2/3/4/5/6$.

Here is one example of an event:

$$E = \{\text{Rolling an odd number}\}.$$

The event space $\Omega = 1, 2, 3, 4, 5, 6$. So the probability of rolling an odd number is

$$\frac{3}{6} = \frac{1}{2}.$$

□

Example 7. What is the probability of rolling two 6 consecutively in two rolls?

Solution: The probability of rolling a 6 is $\frac{1}{6}$. Then rolling two 6 consecutively in two rolls is $\frac{1}{6} \times \frac{1}{6}$. □

Example 8. What is the probability of rolling two 6 consecutively?

Solution: Note that this example is different from the previous example, we allow for any number of rolls. Let P be the required probability,

$$\begin{aligned}P = \sum P(\text{Roll two times, and got two 6}) + P(\text{Roll 3 times, and got two 6 at the last two rolls}) \\ + P(\text{Roll 4 times, and got two 6 at the last two rolls}) \\ + P(\text{Roll 5 times, and got two 6 at the last two rolls})\end{aligned}$$

The probability of rolling n times and got no 6 is $(\frac{5}{6})^n$. (WHY?) Thus

$$P = \frac{1}{6}\frac{1}{6}\sum_{i=0}^{\infty}\left(\frac{5}{6}\right)^i = \frac{1}{6}\frac{1}{6}6 = \frac{1}{6} \tag{20}$$

□

Theorem 1

The probability P of events any E and F in a samples space Ω satisfy:

1. $P(E) \geq 0$, $P(\Omega) = 1$
2. if $E \subset F \subset \Omega$, then $P(E) \leq P(F)$,
3. if $E \cap F = \varnothing$, then $P(E \cup F) = P(E) + P(F)$,
4. $P(E^c) = 1 - P(E)$.

Remark 1

Notice that this is a theorem, not a definition, most of them can be proved from fundamental principle, that the probability satisfies countable additivity, the non-negative of probability measure and the whole space has probability 1. For example, the last item can be proved by rearranging

$$1 = P(\Omega) = P(E^c) + P(E). \tag{21}$$

Example 9. If you roll a dice twice, what is the probability of rolling a sum greater than 2?
Solution: It is not easy to think about what kind of combinations give sum $x \geq 3$. It is certainly doable but requires effort. We can compute it another way,

$$\begin{aligned} P(\text{Rolling a sum } x \geq 3) &= 1 - P(\text{Rolling a sum } x < 3) \\ &= 1 - \underbrace{P(\text{Rolling a sum } x = 1)}_{=0} - P(\text{Rolling a sum } x = 2) \\ &= 1 - P(\text{Rolling a sum } x = 2) \\ &= 1 - \frac{1}{6}\frac{1}{6} \end{aligned} \tag{22}$$

The complement is especially useful in transforming some "$\geq$" into a few "$=$". □

Another important observation from Set Theory is given below.

Theorem 2

For any $E, F \subset \Omega$,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F). \tag{23}$$

This theorem is easy to visualize by drawing the set E and F. Notice that when E and F are disjoint, $P(E \cap F) = 0$ and we recover item 3 in Theorem (1).

Definition 14: Expected Value.

Let X be a random variable. And let x be the possible value of X with probability

$m(x)$. Then the expected value of X is

$$E(X) = \sum_{x \in \Omega} x\, m(x) \tag{24}$$

Example 10. What is the expected value of rolling a fair dice?
Solution: The expected value of rolling a fair dice is

$$E(X) = \sum_{i=1}^{6} i\, \frac{1}{6} = 3.5. \tag{25}$$

The expected value is the average. $\square$

Example 11. What is the expected value of rolling a fair dice until getting a 6? $\square$